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Analysis of the three-dimensional time-dependent Landau–Ginzburg equation and its solutions

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Abstract. The time-dependent Landau–Ginzburg equation of the form $\eta_t + D\Delta\eta = P(\eta)$ where Δ is the three-dimensional Laplacian operator and $P(\eta)$ is an odd-order polynomial up to fifth order, has been used to describe the kinetics of non-conservative order parameters η at and near criticality. The symmetry reduction method has been applied to solve this equation when $P(\eta)$ is given by $P(\eta) = a + b\eta + c\eta^3 + d\eta^5$. This can be used to model both first- and second-order phase transitions which take place with $(a \neq 0)$ or without $(a = 0)$ an external field. When either $d \neq 0$ or $c \neq 0$, two general cases have been found: (i) $a = b = 0$ or $a = b = d = 0$, where the symmetry group involves translations in (3+1)-dimensional spacetime, rotations in three-dimensional space and a dilation; (ii) otherwise, where the symmetry group involves translations in (3+1)-dimensional spacetime and rotations in three-dimensional space. All the reductions to ODE and the corresponding symmetry variables have been derived. The Painlevé-type reduced ODE have been solved exactly while the remaining ones can be analysed numerically or approximately. Physical properties of the obtained solutions have been discussed, including their energies.

1. Introduction

Landau's theory of phase transitions (Landau and Lifshitz 1980) is based on the assumption that the thermodynamic potential G_0 of a near-critical system is expandable in a power series in a symmetry invariant η called the order parameter. For scalar order parameters we have

$$G_0 = \sum_{k=n}^{p(>n)} A_k \eta^k \tag{1}$$

with $A_2 \cong \alpha(T - T_c)$ sufficiently close to T_c. This mean-field approach can be used to describe a variety of second-order (for $p = 4$ or $p = 6$, $A_3 = 0$ and $A_4 > 0$) and first-order (for either $p = 6$, $A_3 = 0$, $A_4 < 0$ or $p = 4$ and $A_3 \neq 0$) phase transitions; both temperature induced (for $n = 2$) and field induced (for $n = 1$). Minimisation of G_0 with respect to η yields the average value of η, $\langle\eta\rangle$, which is zero above the critical temperature and non-zero below it. In the absence of external fields, the second-order transition occurs at $T = T_c$ while the first-order transition occurs at $T_c^* = T_c + A_4^2/4\alpha A_6$. In the latter case thermal hysteresis arises between $T_s \equiv T_c + A_4^2/3\alpha A_6$ and T_c. The mean-field analysis of transitions involving externally applied fields can be found in Shimizu

(1982). It is worth noting the presence of double hysteresis loops in first-order field-induced transitions and single hystersis loops in second-order field-induced transitions.

Aharony (1983) has pointed out that the polynomial of (1) representing G_0 is very useful in studying multicritical points such as, e.g. the tricritical point occurring for $A_1 = A_2 = A_4 = 0$. We assume that at least one of the coefficients A_4 or A_6 is non-zero.

The appearance of a non-zero order parameter is a manifestation of a broken symmetry that may, in general, be of one of the following four types (or combinations thereof): (i) translational; (ii) rotational; (iii) time-reversal and (iv) gauge-invariance (Anderson 1984). Broken symmetries not only lead to mean fields (equilibrium phases), but also to lower-dimensional defect structures such as domain walls, vortices, disclination points, dislocations, grain boundaries, etc. Their analytical form is usually found by minimising G_0, to which a term resulting from the coarse-graining of intersite interactions has been added. For scalar order parameters the simplest such invariant is proportional to $(\nabla\eta)^2$, so that the thermodynamic functional adopted for this paper is

$$G = \int \mathrm{d}^3x[h\eta + A_2\eta^2 + A_4\eta^4 + A_6\eta^6 + \tfrac{1}{2}\delta(\nabla\eta)^2] \tag{2}$$

in three-dimensional space.

In a more general context, the coefficient δ may be regarded as a bilinear form δ_{ij} which couples with $\partial^2\eta/\partial x_i\partial x_j$. If δ_{ij} is positive definite and diagonal, then the substitution $x_i = \sqrt{\delta_{ii}}\, x_i'$ transforms δ_{ij} into an identity matrix. This explains our rationale for treating δ as a scalar coefficient.

Dynamical aspects of critical phenomena such as relaxation times, decay rates, response functions, transport coefficients, etc, (Hohenberg and Halperin 1977) are of much interest and are studied by assuming that $\eta = \eta(x, t)$ where t is the time variable. Time evolution of the order parameter towards its thermodynamic equilibrium can be derived through a Markovian master equation for the probability distribution $P(\eta, t)$ whereby (Metiu *et al* 1976)

$$\tau\frac{\partial}{\partial t}P(\eta, t) = \sum_{\delta\eta}[W(\eta - \delta\eta, \eta)P(\eta - \delta\eta, t) - W(\eta, \eta + \delta\eta)P(\eta, t)] \tag{3}$$

and the transition amplitude W is Gaussian distributed with a Boltzmann weight factor

$$W(\eta, \eta + \delta\eta) = W_0 \exp\left(-\int \frac{[\delta\eta(x)]^2}{\Delta}\,dx\right)\exp\left(-\frac{\beta}{2}[G(\eta + \delta\eta) - G(\eta)]\right) \tag{4}$$

where $\beta = (kT)^{-1}$. This yields the following integro-differential equation of motion for η (Metiu *et al* 1976):

$$\frac{\partial\eta}{\partial t} = -\frac{\beta\Gamma}{2}\frac{\delta G}{\delta\eta} \tag{5}$$

where $\Gamma > 0$ is the so-called Landau–Khalatnikov damping function which sets the timescale of the relaxation process. If η is non-conserved globally (e.g. the deformation modes in structural phase transitions), then (5) is reduced to the equation for the most probable path, namely

$$\frac{\partial \eta}{\partial t}+\nabla^2\eta = a+b\eta+c\eta^3+d\eta^5 \tag{6}$$

where we have rescaled the time variable according to $t\to -\frac{1}{2}\beta\Gamma\delta t$ and $a=h/\delta$, $b=2A_2/\delta$, $c=4A_4/\delta$ and $d=6A_6/\delta$. Equation (6) is called the time-dependent Landau-Ginzburg equation (TDLG) and describes diffusive relaxation in critical systems (Metiu *et al* 1976, Gunton and Droz 1983, Gunton *et al* 1983, Koch 1984). Applications of this equation cover a broad range of systems including liquid crystals (Kawasaki and Ohta 1982), ferroelectrics (Gordon 1983, Ishibashi and Suzuki 1984), structural phase transitions (Parliński and Zielinski 1981), etc. Similar equations have been applied to uniaxial ferromagnets (Khan 1986, Winternitz *et al* 1988, 1989a, b), critical liquids (Cahn and Hilliard 1958), to the kinetics of chemical reactions (Kuramoto 1984), where it is referred to as a reaction-diffusion equation, and to biological processes (Belintsev *et al* 1978). A list of further applications to such disparate areas as genetics, combustion processes, neuron physics and even transmission line problems can be found in the review paper by Fife (1978) and in the paper by Aronson and Weinberger (1975).

Moreover, special cases when $a=0$ are encountered in nonlinear scalar field theories (Jackiw 1977, Winternitz *et al* 1987). In the framework of these theories the nonlinear wave equation is solved on bounded or unbounded domains. When one deals with either real time-independent fields or with complex time-dependent fields where the spatial part is real and time dependence is harmonic, then it is natural to define solutions on the whole $(3+1)$-dimensional spacetime and assume that they must be twice differentiable, positive and should vanish on the boundary. It so happens that solving our problem (TDLG equation), we encounter very similar equations. Formal similarity allows us to adopt some of the field theoretic results.

Each of the particular applications dictates its own limitations. In population genetics, for example, the possible values of the solutions are limited to the $\langle 0,1\rangle$ interval. In field theory, solutions must be continuous in the entire R^N space. In other models this requirement can be changed to an integrability condition and in yet another, solutions have to be bounded at infinity. A vast majority of papers concerned with TDLG or similar equations treat these equations in $(1+1)$-dimensional spacetime.

The main purpose of this paper is to analyse the TDLG equation (6) in $(3+1)$-dimensional spacetime in the context of the kinetics of phase transitions. This is a continuation of an earlier paper (Skierski *et al* 1988) where we presented the results of the symmetry reduction method applied to the TDLG equation. In the present paper we intend to focus on the type and form of solutions of the reduced ordinary differential equations (ODE).

The crucial questions related to this type of analysis address such aspects as the symmetry of the allowed solutions, the rate of phase nucleation, phase separation kinetics, the energy density required to create a particular solution, dynamics of domain walls, the symmetries of lower-dimensional defect structures, the role of dimensionality, etc.

A very brief outline of the methods of calculation (symmetry reduction and Painlevé test) and their results is presented in the next section since a more complete presentation can be found elsewhere in the literature (e.g. Olver 1986, Ablowitz *et al* 1980). Discussion of the manifold of solutions and of the form of symmetry variables then follows. The main part of the paper consists of an analysis of the obtained ODE and their solutions. A subsequent section reviews the results of the stability analysis for various solutions. The final section is concerned with the physical interpretation including the energies of solutions.

2. The methods of calculation

There are two distinct methods applied in the analysis of the TDLG equation. First, the method of symmetry reduction has been used to find all the possible reductions from the given PDE to ODE, consistent with infinitesimal symmetry conditions. Second, for the obtained ODE the Painlevé technique has been employed to determine which of them could be integrated analytically.

2.1. The symmetry reduction method

In the first step, we look for the maximal Lie group of local transformations acting in the space of independent and dependent variables in such a way that the second prolongation of an infinitesimal generator applied to the equation (in our case, equation (6)), should be zero whenever the equation is satisfied. Since no constraints on $\partial\eta/\partial x_i$ and $\partial^2\eta/\partial x_i^2$ $(1 \leq i \leq 3)$ are imposed, the coefficients standing by these derivatives should be zero. This leads to an overdetermined system of equations for the coefficients of infinitesimal generators which is called a system of determining equations. This entire step is completely algorithmic and, in fact, was done using a symbolic computer program written in the MACSYMA symbolic language by Champagne and Winternitz (1985). A system of 18 determining equations was obtained. In our case all the coefficients of infinitesimal generators have been obtained explicitly. Thus, as a result, the following nine generators were obtained for three different Lie algebras:

$$P_i = \frac{\partial}{\partial x_i} \qquad i = 0, 1, 2, 3 \tag{7a}$$

$$L_i = \varepsilon_{ijk} x_k P_j \qquad i, j, k = 1, 2, 3 \tag{7b}$$

$$D_\alpha = x_i P_i + 2x_0 P_0 - \frac{1}{\alpha}\eta\frac{\partial}{\partial\eta} \qquad \alpha = 1, 2. \tag{7c}$$

The commutation relations for these operators are as follows:

$$[L_i, L_k] = \varepsilon_{ikl} L_l \qquad [P_i, P_j] = 0 \qquad [L_i, P_0] = 0 \tag{8a}$$

$$[L_i, P_k] = \varepsilon_{ikl} P_l \qquad [D_\alpha, P_i] = -P_i \qquad [D_\alpha, P_0] = -2P_0 \qquad [D_\alpha, L_i] = 0 \tag{8b}$$

where $i, j = 1, 2, 3$ and $\alpha = 1, 2$.

These operators represent time (P_t) and space translations (P_x, P_y, P_z), spatial rotations $(L_1 = L_{zy}, L_2 = L_{xz}, L_3 = L_{yx})$ and dilations (D_1, D_2), respectively. We have used the notation $(x_0, x_1, x_2, x_3) = (t, x, y, z)$. Note that in (8) the commutator $[D_1, D_2]$ has not been listed since these two operators never appear in the same algebra.

The first algebra is generated by

$$\{P_t, P_x, P_y, P_z, L_{yx}, L_{xz}, L_{zy}\} = e(3) \oplus \{P_t\}$$

where $e(3)$ denotes the algebra corresponding to the Euclidean group consisting of translations and rotations in three-dimensional space. The algebra $e(3)\oplus\{P_t\}$ corresponds to the case when one of the following conditions is satisfied by (6):

(i) $d \neq 0$ and at least one of a, b and c is non-zero

(ii) $d = 0$ and $c \neq 0$ and at least one of a and b is non-zero.

This means that second-order (ii) or first-order (i) spontaneous or field-induced phase transitions are included and all the points on the phase diagram are allowed except for the critical or multicritical points.

The second algebra is generated by

$$\{P_t, P_x, P_y, P_z, L_{yx}, L_{xz}, L_{zy}, D_1\} = (e(3) \oplus \{P_t\}) \oplus \{D_1\}$$

and it corresponds to the case when $c \neq 0$ and $a = b = d = 0$ which means that only the critical point for spontaneous second-order phase transitions is allowed.

The third algebra is generated by

$$\{P_t, P_x, P_y, P_z, L_{yx}, L_{xz}, L_{zy}\quad, D_2\} = (e(3) \oplus \{P_t\}) \oplus \{D_2\}$$

and it corresponds to the case when $d \neq 0$ and $a = b = c = 0$, which means that only the tricritical point for spontaneous phase transitions is allowed.

In the next step we find representative subgroups of all the classes of the adjoint subgroups. The main operations of this algorithm were described in the sequence of papers by Patera *et al* (1974, 1975, 1976a, b, c, 1977). Following this algorithm, we found all such subgroups having generic orbits of codimension one. The corresponding subalgebras are systems of first-order differential operators. For each subalgebra, applying these operators to arbitrary functions of x and η, we obtain a system of first-order PDE which can be easily solved by the method of characteristics. Eliminating the unknown function η from the system of first integrals of these first-order PDE, we obtain new independent variables $\xi(x, t)$ called symmetry variables. This step is crucial in our method since the symmetry variables are invariants of the assumed subgroup. Next, from this system of first integrals we derive the unknown function η as a function of ξ and we obtain

$$\eta = \rho(t, x, y, z) f(\xi(t, x, y, z)) \tag{9}$$

where ρ and ξ are functions of x, y, z and t given by the symmetry of the problem, and $f(\xi)$ is required to satisfy an ODE. Substituting (9) into (6), we reduce the TDLG to an ODE of first or second order. The results of this procedure are summarised in table 1. This procedure allows for a systematic classification of the reduced equations and their solutions from a group theoretical point of view.

For a more comprehensive exposition of the symmetry reduction method the reader is referred to, for example, Olver (1986).

2.2. *The Painlevé test*

As stated by Ablowitz *et al* (1980), an ODE has the Painlevé property if the family of its solutions has no movable critical points (branch points or essential singularities). It has been conjectured that solutions of a second-order ODE having this property can be expressed in terms of elementary functions, elliptic functions or the six Painlevé transcendents. Thus, the knowledge of whether or not a given ODE satisfies the Painlevé test will assist in searching for its analytic solutions. Ablowitz *et al* (1980) gave an algorithm which verifies the necessary conditions for any ODE to have the Painlevé property. In the first step of this procedure the behaviour of solutions in the vicinity of a singular point is investigated. The dominant exponent in a series expansion must be a negative integer, otherwise it is a critical point. In the second step the relationship between the dominant exponent p and some higher exponents $p+r$ where $r>0$ is investigated. Obviously, to avoid critical points, r must be a positive integer. Finally,

Table 1. The results of symmetry reduction for the TDLG equation (6), in the general case of arbitrary values for the parameters a, b, c and d. Here, we denote $\alpha=-\frac{1}{2}$ for the $a=b=d=0$ case and $\alpha=-\frac{1}{4}$ for the $a=b=c=0$ case.

Case	ξ	ρ	ODE	Painlevé?
1	t	1	$\frac{df}{d\xi}=a+bf+cf^3+df^5$	First order
2	x	1	$\frac{d^2f}{d\xi^2}=a+bf+cf^3+df^5$	Yes for $a=0$ or $d=0$
3	$v(x+vt)$	1	$\frac{d^2f}{d\xi^2}+\frac{df}{d\xi}=\frac{1}{v^2}(a+bf+cf^3+df^5)$	No
4	$\sqrt{x^2+y^2}$	1	$\frac{d^2f}{d\xi^2}+\frac{1}{\xi}\frac{df}{d\xi}=a+bf+cf^3+df^5$	No
5	$\sqrt{x^2+y^2+z^2}$	1	$\frac{d^2f}{d\xi^2}+\frac{2}{\xi}\frac{df}{d\xi}=a+bf+cf^3+df^5$	Yes for $a=b=c=0$
6a	$\tan^{-1}y/x$	$(x^2+y^2)^\alpha$	$\frac{d^2f}{d\xi^2}+f-cf^3=0$	Yes
6b	$\frac{1}{2}\tan^{-1}y/x$	$(x^2+y^2)^\alpha$	$\frac{d^2f}{d\xi^2}+f-df^5=0$	Yes
7a	$-\frac{2B}{B^2+1}[\tan^{-1}(y/x)+\frac{1}{2}B\log(x^2+y^2)]$	$\left(\frac{4B^2}{(B^2+1)(x^2+y^2)}\right)^{-\alpha}$	$\frac{d^2f}{d\xi^2}+\frac{df}{d\xi}+\frac{B^2+1}{4B^2}f-cf^3=0$	Yes for $B=\pm 3\mathrm{i}$
7b	$-\frac{2B}{B^2+1}[\tan^{-1}(y/x)-\frac{1}{2}B\log(x^2+y^2)]$	$\left(\frac{4B^2}{(B^2+1)(x^2+y^2)}\right)^{-\alpha}$	$\frac{d^2f}{d\xi^2}+\frac{df}{d\xi}+\frac{B^2+1}{4B^2}f-df^5=0$	Yes for $B=\pm 2\mathrm{i}$
8a	$\cot^{-1}\frac{z}{\sqrt{x^2+y^2}}$	$(x^2+y^2+z^2)^\alpha$	$\frac{d^2f}{d\xi^2}+\cot(\xi)\frac{df}{d\xi}-cf^3=0$	No
8b	$\cot^{-1}\frac{z}{\sqrt{x^2+y^2}}$	$(x^2+y^2+z^2)^\alpha$	$\frac{d^2f}{d\xi^2}+\cot(\xi)\frac{df}{d\xi}+\frac{1}{4}f-df^5=0$	No
9a	$t/2x^2$	$x^{2\alpha}$	$\xi^2\frac{d^2f}{d\xi^2}+(5\xi^2+\frac{1}{2})\frac{df}{d\xi}+2f-cf^3=0$	No
9b	$t/2x^2$	$x^{2\alpha}$	$\xi^2\frac{d^2f}{d\xi^2}+(4\xi^2+\frac{1}{2})\frac{df}{d\xi}+\frac{3}{4}f-df^5=0$	No
10a	$t/[2(y^2+z^2)]$	$(y^2+z^2)^\alpha$	$\xi^2\frac{d^2f}{d\xi^2}+(4\xi+\frac{1}{2})\frac{df}{d\xi}+f-cf^3=0$	No
10b	$t/[2(y^2+z^2)]$	$(y^2+z^2)^\alpha$	$\xi^2\frac{d^2f}{d\xi^2}+(3\xi+\frac{1}{2})\frac{df}{d\xi}+\frac{1}{4}f-df^5=0$	No
11a	$t/(x^2+y^2+z^2)$	$(x^2+y^2+z^2)^\alpha$	$\xi^2\frac{d^2f}{d\xi^2}+(6\xi+1)\frac{df}{d\xi}-cf^3=0$	No
11b	$t/(x^2+y^2+z^2)$	$(x^2+y^2+z^2)^\alpha$	$\xi^2\frac{d^2f}{d\xi^2}+(4\xi+1)\frac{df}{d\xi}-f-df^5=0$	No

the relationships between expansion coefficients are analysed. The general solution of an nth-order ODE depends on n arbitrary constants. Since one of them is given by the position of the singularity, there are only $n-1$ independent constants. Due to the nonlinearity of the ODE, we are usually unable to find a recursive relation for all the coefficients. The algorithm discussed here limits the investigation to a certain number of first terms (up to the term indexed by the highest positive integer resonance) whose powers were related to each other in the second step. Therefore, the necessary condition is that these coefficients depend only on $n-1$ constants. It should be emphasised that examples of ODE which pass the Painlevé test and yet possess movable essential singularities are known to exist (Ablowitz *et al* 1980).

The above procedure has been performed by a computer using the program written in the MACSYMA symbolic language by Rand and Winternitz (1986). The results of this test are presented in the last column of table 1. In particular, case 2 and its various special cases can be integrated immediately, multiplying by $\mathrm{d}f/\mathrm{d}\xi$, and we obtain, in general

$$(\mathrm{d}f/\mathrm{d}\xi)^2 = 2\left[a_0 + af + \frac{bf}{2} + \frac{cf^4}{4} + \frac{df^6}{6}\right] \tag{10}$$

which is thus reduced to quadratures, and its solutions can be found in terms of elementary or elliptic functions. For equation 2 the substitution

$$f(\xi) = \sqrt{h(\xi)} \tag{11}$$

gives

$$\frac{\mathrm{d}^2h}{\mathrm{d}\xi^2} = \frac{1}{2h}\left(\frac{\mathrm{d}h}{\mathrm{d}\xi}\right)^2 - \frac{2}{\xi}\frac{\mathrm{d}h}{\mathrm{d}\xi} + 2\mathrm{d}h^3. \tag{12}$$

Following Winternitz *et al* (1988a), case 7a can be transformed to its Painlevé form P VII (Ince 1956) using the substitution

$$f(\xi) = \lambda(\xi)\, W(\eta(\xi)). \tag{13}$$

Finally, for case 7b, we use substitution given by (11) followed by that of (13) to obtain its standard form P XXIX (Ince 1956).

It should be noted that a Painlevé test similar to the one used in this paper has been applied directly to PDE (Weiss *et al* 1983). In fact, Keefe (1986) investigated the integrability of the complex cubic Landau-Ginzburg equation in $(1+1)$ dimensions. It was found that it does not, in general, possess the Painlevé property, except when it corresponds to the integrable nonlinear Schrödinger equation. Therefore, we do not expect that our time-dependent Landau-Ginzburg equation is of Painlevé type in $(3+1)$ dimensions. This is reflected in the fact that the reduced ODE presented in our tables do not have the Painlevé property except for the above-listed special cases.

3. Analysis of the symmetry variables

Solutions of the PDE (6) are specified by appropriate boundary conditions. The applied method limits surfaces on which boundary conditions may be defined to the orbits of subgroups of the symmetry group.

Furthermore, a solution η of the given PDE is related to a solution f of the corresponding ODE through the relation $\eta(x, y, z, t) = \rho(x, y, z, t)\, f(\xi(x, y, z, t))$ where ξ is constant on each orbit.

The following discussion of the geometry of orbits will provide us with an insight into both of these problems. We will also describe the geometry of orbits of lower dimension at the same time. The independent variables used are written in the following order: t, x, y, z. We have found the following eleven distinct cases of symmetry variables.

(i) $\xi = t$. This is a particularly interesting case since t appears only in the first-order derivative. The domain of this symmetry variable is $\mathbb{R}$ and the domain of the PDE is $\mathbb{R}^4$.

(ii) $\xi = x$. This is just a prototype of a more general symmetry variable which can be an arbitrary linear combination of x, y and z. It describes quasi-one-dimensional stationary solutions. The equivalued surfaces for the order parameter are planes orthogonal to ξ; in this particular choice of ξ these are yz planes. The domain of this symmetry variable is $\mathbb{R}$.

(iii) $\xi = x + vt$. This represents a quasi-one-dimensional solution moving with a constant velocity. The equivalued surfaces for the order parameter are planes orthogonal to ξ. The domain of this variable is $\mathbb{R}$.

(iv) $\xi = \sqrt{x^2+y^2} \equiv r$. The equivalued surfaces for the order parameter are families of coaxial cylinders whose axes pass through the origin and are parallel to the z axis. The domain of this symmetry variable is $\mathbb{R}_+$. The z axis contains singularities of the order parameter.

(v) $\xi = \sqrt{x^2+y^2+z^2} \equiv R$. The equivalued surfaces for the order parameter are spheres centred at the origin. The domain of this symmetry variable is $\mathbb{R}_+$. There is a singularity at the origin.

(vi) $\xi = \tan^{-1}(y/x) \equiv \phi$. In cylindrical coordinates ξ represents an angle. For this reason, imposing periodic conditions on the solutions implies that the domain of ξ is $\langle 0, 2\pi \rangle$. Although ξ describes planes parallel to the z axis, the order parameter diminishes radially as the solution moves perpendicularly away from the z axis since f is multiplied by $\rho \sim r^{-1}$ for $d=0$ and by $\rho \sim r^{-1/2}$ for $d \neq 0$.

(vii) $\xi = \tan^{-1}(y/x) + \frac{1}{2}B\log(x^2+y^2)$. In cylindrical coordinates ξ becomes $\xi = \phi + B\log r$. Although ξ describes logarithmic spiral surfaces parallel to the z axis, the order parameter diminishes radially as the solution moves away from the origin since f is multiplied by $\rho \sim r^{-1}$ for $d=0$ and by $\rho \sim r^{-1/2}$ for $d \neq 0$. With periodic boundary conditions imposed on the solution, the domain of the symmetry variable is the proper interval $\langle 0, 2\pi \rangle$.

(viii) $\xi = \cot^{-1}(z/\sqrt{x^2+y^2}) \equiv \theta$. In spherical coordinates, ξ corresponds to the angle between the z axis and the position vector. The domain of ξ is therefore $(0, \pi)$. Although ξ describes cones symmetric with respect to the z axis where the origin of the coordinate system has been excluded, the order parameter diminishes radially as the solution moves away from the origin since f is multiplied by $\rho \sim r^{-1}$ for $d=0$ and by $\rho \sim r^{-1/2}$ for $d \neq 0$.

(ix) $\xi = t/2x^2$. This symmetry variable describes moving planes and the order parameter diminsishes inversely proportionally with the distance from the origin. The domain of ξ is $\mathbb{R}$.

(x) $\xi = t/2(y^2+z^2)$. This symmetry variable describes a family of moving cylinders. The order parameter diminishes radially as the solution moves away from the cylinder's axis. The domain of ξ is $\mathbb{R}$.

(xi) $\xi = t(x^2+y^2+z^2)^{-1} = t/R$. Although ξ describes spheres with radii expanding to infinity as $t \to \infty$, the order parameter diminishes radially as the solution moves away from the origin since f is multiplied by $\rho \sim R^{-1}$ for $d=0$ and $\rho \sim R^{-1/2}$ for $d \neq 0$. This symmetry variable is defined on the entire $\mathbb{R}$. The origin is in an orbit of a smaller dimension.

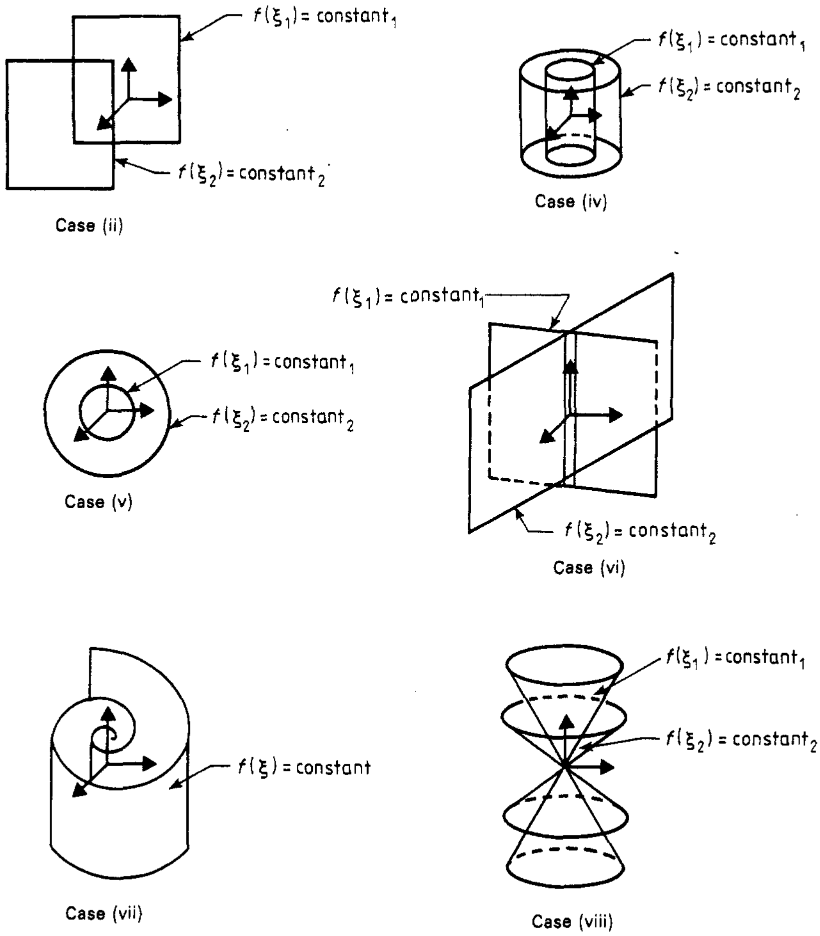

Figure 1. A schematic illustration of the equivalued surfaces for the order parameter in cases (ii)-(viii) referred to in § 3.

In figure 1 we have schematically illustrated the surfaces on which ξ is constant in cases (ii)-(viii). Case (i) is trivial, cases (iii) and (ix) can be illustrated jointly with case (ii), case (x) with case (iv) and case (xi) with case (v).

4. Solutions of the reduced ODE

Since most of the considered ODE have continuous coefficients in the domain, we infer that for an arbitrary initial condition there exists a unique solution to the equation. However, in some applications we must deal with boundary conditions and in such cases there are very few facts known about the existence and uniqueness of solutions. See Winternitz *et al* (1988) for a discussion on the physical interpretation of the boundary conditions and the method of symmetry reduction.

Case 1 from table 1 and its special cases, are readily integrable provided we know at least one root of the polynomial on the right-hand side. Unfortunately, there are no analytical methods for finding roots of a general quintic polynomial. On the other hand, there are many computer programs for finding at least one root of a polynomial (IMSL library) which enable one to decompose the polynomial into a product of first-degree or second-degree polynomials. We first consider the general form of case 1

$$\frac{\mathrm{d}f}{\mathrm{d}\xi}=a+bf+cf^3+df^5. \tag{14}$$

The analysis of its solutions depends on the type of roots of the polynomial on the right-hand side of (14) and is elementary. All the individual cases are integrable by the fractional parts method in terms of elementary functions. In most cases the obtained solutions cannot be inverted, so that results remain in implicit form. However, the asymptotic behaviour of the solutions can be found immediately from the form of the polynomial. Roots of the polynomial correspond to singular solutions which are locally stable and which occur when the polynomial changes sign from positive to negative. They are attractive from one side and repulsive from the other when the root is of even order. This will be elaborated on in the next section. In addition, since this equation is autonomous, if $f(\xi)$ is a solution, then $f(\xi+\xi_0)$ is also a solution, for any ξ_0. It is apparent that solutions interpolate between the neighbouring stable roots (homogeneous phases).

As special cases we can consider

$$\frac{\mathrm{d}f}{\mathrm{d}\xi}=cf^3 \tag{15}$$

whose explicit solution is

$$f(\xi)=[2c(\xi-\xi_0)]^{-1/2} \tag{16}$$

and the equation

$$\frac{\mathrm{d}f}{\mathrm{d}\xi}=df^5 \tag{17}$$

whose explicit solution is

$$f(\xi)=[4d(\xi-\xi_0)]^{-1/4}. \tag{18}$$

Both of them have singular points at $\xi=\xi_0$ and tend asymptotically to zero as $\xi\to\infty$.

Case 2 from table 1

$$\frac{\mathrm{d}^2f}{\mathrm{d}\xi^2}=a+bf+cf^3+df^5 \tag{19}$$

has recently received considerable attention and all of its exact solutions have been found for $a=0$ and $d\neq 0$ (Winternitz *et al* 1987), for $a=d=0$ (Winternitz *et al* 1988) and for $a\neq 0$, $d=0$ (Winternitz *et al* 1989a). Solutions of the general case with a and d non-zero have not been found explicitly. However, integrating (19) yields first (10), and a subsequent integration yields

$$\pm(\xi-\xi_0)=\frac{1}{\sqrt{2}}\int \mathrm{d}f\left(af+\frac{bf^2}{2}+\frac{cf^4}{4}+\frac{df^6}{6}+a_0\right)^{-1/2} \tag{20}$$

where ξ_0 and a_0 are integration constants. It is easy to see that the family of solutions of this equation includes both complex and real functions and the former have to be discarded on physical grounds in the present context. Among the latter, one finds solutions with and without singular points. Solutions with singular points can be interpreted as defect structures of the order parameter. Solutions which do not have singular points may represent extended modes (i.e. periodic) and in particular may be trigonometric, but in general they are elliptic functions. They may also represent localised modes and in this category one finds algebraic functions, bumps (non-topological solitons) and kinks (topological solitons). The occurrence of the various types of solutions is determined solely by the values of the parameters a, b, c, d and the integration constant a_0. In figure 2 we have illustrated this, showing the bracketed term in the integrand of (20) as a function of f for a special choice of parameters and the ensuing location of particular solutions which interpolate between classical turning points. It turns out that:

(a) when the line joining two turning points is above the curve describing the bracketed term in the integrand as a function of f, then the solution is complex;

(b) when there is only one turning point (or none), the solution has singular points;

(c) when both turning points are single roots of the polynomial in the integrand the solution is periodic and bounded;

(d) When one of the turning points is a double (or multiple in general) root of the polynomial, then the solution is a bump;

(e) when both of the turning points are double (or multiple in general) roots of the polynomial, then the solution is a kink.

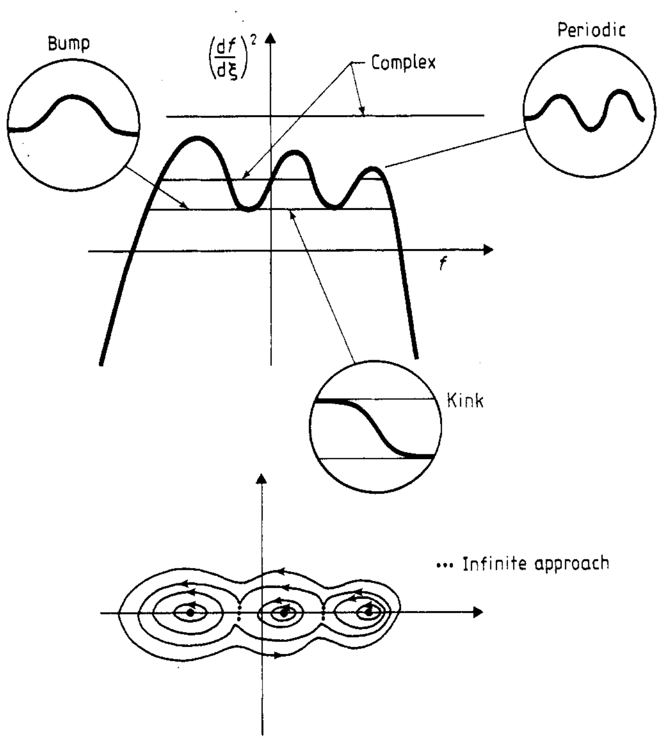

Figure 2. A typical situation encountered for (19) and its solutions.

For explicit forms of these various solutions the reader is referred to the papers of Winternitz et al (1987, 1988, 1989a).

A special case of (19) with $a=b=d=0$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2}=cf^3 \tag{21}$$

can be integrated once to give

$$\left(\frac{\mathrm{d}f}{\mathrm{d}\xi}\right)^2=\tfrac{1}{2}c(f^4+r_0) \tag{22}$$

where r_0 is an integration constant. For $c>0$ all the solutions are either complex or real and have singular points (see Winternitz *et al* 1988). For $c<0$, on the other hand, when $r_0>0$ the solutions are complex while for $r_0\leqslant 0$, they are real bounded and periodic and can be expressed as

$$f=|r_0|^{1/4}\,\mathrm{cn}\left(|c^2 r_0|^{1/4}(\xi-\xi_0),\frac{1}{\sqrt{2}}\right). \tag{23}$$

Another special case with $a=b=c=0$ is

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2}=df^5 \tag{24}$$

integrating once and substituting $f=\sqrt{h}$, it yields

$$\left(\frac{\mathrm{d}h}{\mathrm{d}\xi}\right)^2=\tfrac{4}{3}d(h^4+r_0 h) \tag{25}$$

where r_0 is an integration constant. Again, its solutions have been analysed by Winternitz *et al* (1987) and for $d>0$ they are either complex or real with poles. For $d<0$ and $r_0>0$ the solutions are complex while for $r_0\leqslant 0$ they are real bounded and periodic and can be expressed as

$$f=\left\{\frac{|r_0|^{1/3}}{1+\sqrt{3}}\,\frac{1-\mathrm{cn}(\sqrt[4]{27}\,\frac{1}{2}|d|^{-1/2}|r_0|^{-1/3}(\xi-\xi_0),\frac{1}{2}\sqrt{2-\sqrt{3}})}{1+\mathrm{cn}(\sqrt[4]{27}\,\frac{1}{2}|d|^{-1/2}|r_0|^{-1/3}(\xi-\xi_0),\frac{1}{2}\sqrt{2-\sqrt{3}})}\right\}^{1/2}. \tag{26}$$

For these types of equations Ehrmann (1957) and independently Fucik and Lovicar (1975) proved that for the boundary problem on a bounded domain there is an infinite number of distinct solutions when it is assumed only that $d<0$ or $d=0$ and $c<0$.

Case 6a of table 1

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2}+f+\varepsilon f^3=0 \tag{27}$$

can be integrated once to give

$$\left(\frac{\mathrm{d}f}{\mathrm{d}\xi}\right)^2=-\tfrac{1}{2}\varepsilon(f^4+2\varepsilon f^2-r_0) \tag{28}$$

and its solutions can be found in Winternitz *et al* (1988). For the boundary conditions $f(0)=f(2\pi)$ and $f'(0)=f'(2\pi)$, and for $\varepsilon=1$ and $r_0>0$ solutions are complex while for $r_0<0$ they can be expressed as

$$f_n(\xi)=\sqrt{\sqrt{1+r_n}-1}\,\mathrm{cn}\left[[4(1+r_n)]^{1/4}(\xi-\xi_0),\tfrac{1}{2}-\left(\frac{1}{2\sqrt{1+r_n}}\right)^{1/2}\right] \tag{29}$$

where the integration constant r_n is discrete since it must satisfy the transcendental equation

$$K\left[\left(\frac{1}{2}-\frac{1}{2\sqrt{1+r_n}}\right)^{1/2}\right]=\frac{\pi(1+r_n)^{1/4}}{\sqrt{2}n} \qquad n=2,3,4,\ldots \tag{30}$$

for $\varepsilon=-1$ there is only one real periodic and bounded solution, namely

$$f_1(\xi)=(1-\sqrt{1+r_1})^{1/2}\,\mathrm{sn}\left[(1+\sqrt{1+r_1})^{1/2}(\xi-\xi_0),\left(\frac{1-\sqrt{1+r_1}}{1+\sqrt{1+r_1}}\right)^{1/2}\right] \tag{31}$$

where $-1<r_1<0$ and r_1 is such that

$$\frac{\pi}{2}(1+\sqrt{1+r_1})^{1/2}=K\left[\left(\frac{1-\sqrt{1+r_1}}{1+\sqrt{1+r_1}}\right)^{1/2}\right]. \tag{32}$$

Case 6b of table 1

$$\frac{\mathrm{d}^2f}{\mathrm{d}\xi^2}+f+\varepsilon f^5=0 \tag{33}$$

can be integrated by substituting $f=\sqrt{h}$ which yields

$$\left(\frac{\mathrm{d}h}{\mathrm{d}\xi}\right)^2=(-\tfrac{4}{3}\varepsilon)(h^4+3\varepsilon h^2+r_0h). \tag{34}$$

Again, when $\varepsilon=1$ and $r_0>0$, the solutions are complex, while for $r_0<0$ they are bounded real and periodic. A complete analysis of the solutions to (34) has been given by Winternitz *et al* (1987) and in general we can write it as

$$f=\left(\frac{z-1}{\mu z+\nu}\right)^{1/2} \tag{35}$$

where z is either $\mathrm{sn}(A_1\xi,k)$ or $\mathrm{cn}(A_2\xi,k)$ and μ, ν, A_1, A_2, A_3 and k can readily be found in terms of the roots of the polynomial in (34). Solutions with $\varepsilon=-1$ are mainly real with poles or complex, but for some choices of r_0 ($|r_0|\leqslant 2\sqrt{2}$) may also involve bounded elliptic functions or bumps. The difference between this case and the previous one is in the periodicity condition on the Jacobi elliptic parameter k which is now

$$4K(k)/A_i=n\pi \qquad i=1,2,3 \tag{36}$$

and k depends on r_0.

Case 4 of table 1 is

$$\frac{\mathrm{d}^2f}{\mathrm{d}\xi^2}+\frac{1}{\xi}\frac{\mathrm{d}f}{\mathrm{d}\xi}=a+bf+cf^3+df^5. \tag{37}$$

In the special case of $a=0$, it has been investigated in a series of papers listed in the bibliography of Anderson and Derrick (1970). The existence of particle-like solutions was proved by Berestycki *et al* (1983). It is worth mentioning that they assumed that its solutions do not have a singularity at the origin, which means that $\lim_{\xi\to 0^+}\mathrm{d}f/\mathrm{d}\xi=0$. The assumptions of the main theorem of their paper are fulfilled when $a=0$ and $b>0$ and one of the following conditions is satisfied: (i) $d<0$; (ii) $d>0$ and $c<-\sqrt{16bd/3}$.

Special cases of case 4, namely

$$\frac{\mathrm{d}^2f}{\mathrm{d}\xi^2}+\frac{1}{\xi}\frac{\mathrm{d}f}{\mathrm{d}\xi}=cf^3 \tag{38}$$

and

$$\frac{d^2f}{d\xi^2}+\frac{1}{\xi}\frac{df}{d\xi}=df^5 \tag{39}$$

have the interesting property that if $f(\xi)$ is a solution then so is $A^{\beta}f(A\xi)$ where $\beta=2/(\nu-1)$; $\nu=3$, 5, respectively, and A is an arbitrary constant. The idea of a numerical algorithm for finding solutions for the argument greater than 1 ($x>1$) is essentially identical to that for the Emden equation discussed below.

Case 5 of table 1

$$\frac{d^2f}{d\xi^2}+\frac{2}{\xi}\frac{df}{d\xi}=a+bf+cf^3+df^5 \tag{40}$$

was investigated in the context of field theory by Berestycki and Lions (1981, 1983). They proved the existence of a unique ground state for $a=0$ and $b>0$ under one of the assumptions: (i) $d=0$ and $c<0$; (ii) $d>0$ and $c<-\sqrt{16bd/3}$. They also showed the existence of an infinite sequence of solutions such that the value of the action goes to infinity as the index of a solution increases.

Special cases of (40), namely

$$\frac{d^2f}{d\xi^2}+\frac{2}{\xi}\frac{df}{d\xi}=cf^3 \tag{41}$$

and

$$\frac{d^2f}{d\xi^2}+\frac{2}{\xi}\frac{df}{d\xi}=df^5 \tag{42}$$

have been discussed at length and (42) has been analytically solved by Winternitz *et al* (1987) and also by Cieciura and Grundland (1984). They are also special cases of the Emden equation. The Emden equation was intensively investigated by astrophysicists in the 1930s. Most of the results can be found in BAAS (1932) and in Davies (1962). They include a description of the asymptotic expansion in the neighbourhood of the origin and a numerical algorithm for the calculation of solutions with larger arguments. For (42) an explicit solution satisfying the condition that $f'(0)=0$ is given by

$$f(\xi)=\left(\frac{3C}{3C^2+\xi^2}\right)^{1/2}. \tag{43}$$

It is especially interesting to note that similar solutions of (40) are unstable for $b>0$.

In general, special types of spherical and cylindrical solutions can be viewed as solutions to

$$\frac{d^2f}{d\xi^2}+\frac{n}{\xi}\frac{df}{d\xi}=bf+cf^3. \tag{44}$$

When $b=0$, a special solution can be found in the form

$$f=\pm\left(\frac{2-n}{c}\right)^{1/2}\xi^{-1}. \tag{45}$$

For small arguments ($\xi\ll\infty$) one can perturb the solutions of (44) around $b=0$ to obtain an approximate series expansion

$$f(\xi)=\pm\left(\frac{2-n}{c}\right)^{1/2}\frac{1}{\xi}+b\left[\frac{A_1}{\xi}+A_2\xi^{2-n}\pm\frac{1}{2(n-1)}\left(\frac{2-n}{c}\right)^{1/2}\xi\right]+\ldots \qquad \text{for} \quad n\neq 1,3 \tag{46}$$

$$f(\xi)=\pm\left(\frac{2-n}{c}\right)^{1/2}\frac{1}{\xi}+b\left(\frac{B_1}{\xi}+B_2\xi\pm\frac{1}{2\sqrt{c}}\xi\ln\xi\right)+\ldots\qquad\text{for } n=1 \tag{47}$$

where the expansion coefficients A_1, A_2 and B_1, B_2 can be easily evaluated. Very recently, Banerjee and Cao (1988) found rational algebraic solutions to a similar equation

$$\frac{d^2f}{d\xi^2}+\frac{n}{\xi}\frac{df}{d\xi}=ef^2+cf^3 \tag{48}$$

as

$$f(\xi)=\left(\frac{4}{3-n}\right)\left(-\frac{e}{c}\right)\left[1+\frac{2}{(3-n)^2}\left(-\frac{e^2}{c}\right)\xi^2\right]^{-1} \tag{49}$$

for $c<0$ and $e>0$. We suspect that their method may be applicable to certain special cases of our equations and intend to pursue this question in the near future.

Finally, Cieciura and Grundland (1984) considered (40) with $a=c=0$; $b<0$ and $d<0$. Using the technique of Kurdgelaidze (1954) they found the solution

$$f(\xi)=[(\lambda/\beta)^{1/4}\xi_0]^{-1/2}\left(\frac{\xi}{\xi_0}\right)^{1/2}Z\left[\omega\left(\frac{\xi}{\xi_0}\right)\right] \tag{50}$$

where λ, β, ξ_0 are real constants, and

$$\omega\left(\frac{\xi}{\xi_0}\right)=\left(\frac{\lambda}{\beta}\right)^{1/4}\ln\left(\frac{\xi}{\xi_0}\right) \tag{51}$$

and Z satisfies the equation

$$\left(\frac{dZ}{d\omega}\right)^2+\alpha Z^2+\tfrac{1}{3}\beta Z^6=C_1 \tag{52}$$

and thus can be expressed by elliptic functions. A special solution with $C_1=0$ takes the Schonster-Emden form:

$$f(\xi)=\left(\frac{3}{\xi_0}\right)^{1/4}\left[1+\left(\frac{\xi}{\xi_0}\right)^2\right]^{-1/2}. \tag{53}$$

For cases 7-11 it is easy to find singular solutions of the reduced ODE. Thus, since $\eta=\rho f(\xi)$, we can generate in this way special solutions of the PDE, (6).

Case 7a of table 1

$$\frac{d^2f}{d\xi^2}+\frac{df}{d\xi}+\frac{B^2+1}{4B^2}f+\varepsilon f^3=0 \tag{54}$$

is of Painlevé type only for $B=3\mathrm{i}$, in which case the general solution can be found in Winternitz *et al* (1988a). Unfortunately, all its solutions are complex except for the one when both integration constants are zero, i.e. $\eta=\pm[c(x^2+y^2)]^{-1/2}$.

The general solution is

$$f(\xi)=\pm\tfrac{1}{2}c_1\exp(-\tfrac{1}{3}\xi)\,\mathrm{sd}[\sqrt{2}\,(c_1\exp(-\tfrac{1}{3}\xi)+c_2),\sqrt{2}] \tag{55}$$

where c_1, c_2 are integration constants and the Jacobi modulus is $k=\sqrt{2}$.

Case 7b of table 1

$$\frac{d^2f}{d\xi^2}+\frac{df}{d\xi}+\frac{B^2+1}{4B^2}f+\varepsilon f^5=0 \tag{56}$$

passes the Painlevé test only for $B=2\mathrm{i}$. Then, its general solution is

$$f(\xi)=\left[-\left(\frac{-3}{8\varepsilon}\right)^{1/2}\exp(-\tfrac{1}{2}\xi)\,W[\exp(-\tfrac{1}{2}\xi)]\right]^{1/2} \tag{57}$$

where W is an arbitrary solution of the equation

$$\dot{W}^2=W^4+CW. \tag{58}$$

All its solutions are complex except the trivial one:

$$\eta=\pm[(-4d)(x^2+y^2)]^{-1/4}.$$

Using the Adams method we have produced plots of numerical solutions to some of these equations. The IMSL package was employed. As the most general example we have used case 4 from table 1 and used a number of initial conditions. It is quite apparent from figure 3 that the solutions are very sensitive to initial conditions. Similar results have been found for the remaining non-Painlevé-type equations.

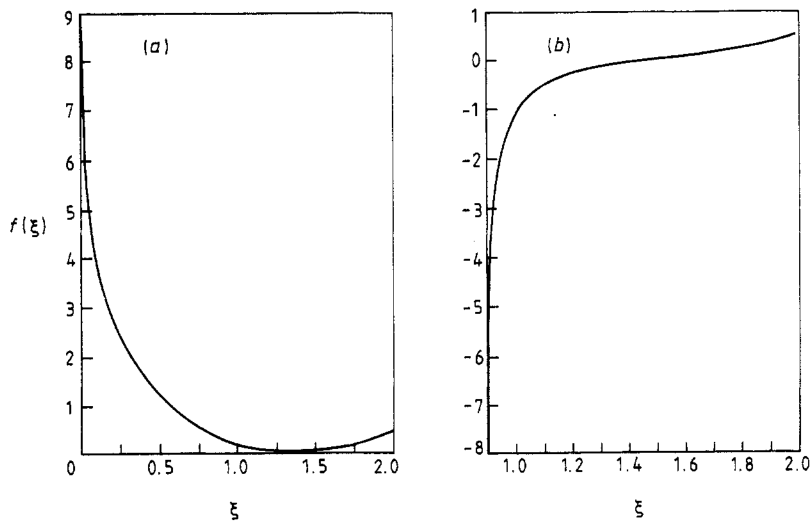

Figure 3. Numerical solutions of case 4 of table 1 for the polynomial roots $f_0=0$, $f_1=1$, $f_2=2$, $f_3=3$, $f_4=4$ and several initial conditions.

6. Stability properties

In the framework of scalar field theories some results concerning stability of time-dependent solutions of (6) were obtained by Berestycki *et al* (1981). For a general problem we consider

$$\frac{\partial\eta}{\partial t}=\Delta\eta+g(\eta) \tag{59}$$

with $\eta(x,0)=\psi(x)$ and $g(\eta)$ must satisfy the condition (i) or (ii) stated below (40). Consider η to be a solution of (37) or (40). It can be regarded as a stationary solution of (59) defined on $S_t=R^N$; $\psi(x,0)$, $x\in S_t$. From the general theorems we know that a solution of (59) exists for arbitrary initial conditions $\psi(x,0)\in C^{2+\nu}(S_t)$ where $S_t=R^N(0,t)$. We shall denote this solution by $u(t,g)$.

Definition. We say that ϕ is stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|\phi - \psi\| < \delta$ for $\psi \in C^{2+\nu}(S_t)$, then:

(i) $t(\psi) = \infty$

(ii) $\|u(t, .) - \phi\| < \varepsilon$ for any $t \in \mathbb{R}_+$.

Here $\|\psi\|$ denotes the absolute value norm in $C^0(\mathbb{R}^N)$. Otherwise, ϕ is called unstable.

Theorem. Let the right-hand sides of (37) and (40) satisfy the conditions (i) and (ii) below (37). Then, any positive, radially symmetric solution η of (37) or (40) which tends to zero at infinity is unstable.

It is important to note that the condition $d > 0$ and $c = -\sqrt{16bd/3}$ corresponds physically to the transition point for first-order phase transitions. Hence, the above theorem confirms our expectations concerning the stability of solutions. It is obvious that under the assumption $a = 0$, (37) and (40) are invariant with respect to the change of the dependent variable $f \to -f$. Therefore, the result remains true for a negative, radially symmetric solution which tends to zero at infinity. Another consequence of this theorem is that solutions of (37) and (40) which are finite at 0 and vanish at infinity can be stable only when they have at least one zero. The stability of (40) in the context of field theory models was also investigated by Anderson and Derrick (1970) but because they considered complex solutions, their results are not very relevant to our considerations.

The stability of solutions of (15) and (17) can be analysed in the same way as the stability of solutions of (14). There are specific results in the literature given by Fife (1978) and Aronson and Weinberger (1975). They showed that there are no bounded stable asymptotic states for $c > 0$ (correspondingly for $d > 0$) and that the trivial solution 0 is the only bounded asymptotic state when $c < 0$ ($d < 0$).

A different approach to the stability of PDE was adopted by Derrick (1964) in the context of nonlinear field theories. A time-independent solution was defined as stable when at a given point the functional derivative of the energy functional (in our case (2)) is equal to zero while the second functional derivative is positive definite at this point. Using such an interpretation, Derrick (1965) showed that (19) written in three spatial dimensions has no stable solutions which are time independent.

Treated as strictly one dimensional, solutions of the reduced ODE may be analysed for stability. However, in our model two independent variables are still present and these solutions are in fact quasi-one-dimensional. A small perturbation will very likely cause dissipation of energy into the directions orthogonal to the propagation axis, resulting in a loss of stability by the solution. This aspect, however, requires further analysis. For now, however, we shall make a few comments about the stability properties of these ODE.

First, the stability of solutions of (14) can be easily found from the graph of the first derivative $\mathrm{d}f/\mathrm{d}\xi$ as a function of f (see figure 4). A singular solution (constant equal to one of the real roots of the polynomial on the right-hand side of (14)) is attractive if the polynomial decreases at this point, is repulsive when the polynomial increases at this point and is repulsive from one side and attractive from the other, otherwise.

Second, for (19) and its special cases, (21), (24), (27) and (33), we may integrate it once and take a square root to obtain

$$\frac{\mathrm{d}f}{\mathrm{d}\xi} = \pm\sqrt{2}\left[af + \frac{bf^2}{2} + \frac{cf^4}{4} + \frac{df^6}{6} + a_0\right]^{1/2}. \tag{60}$$

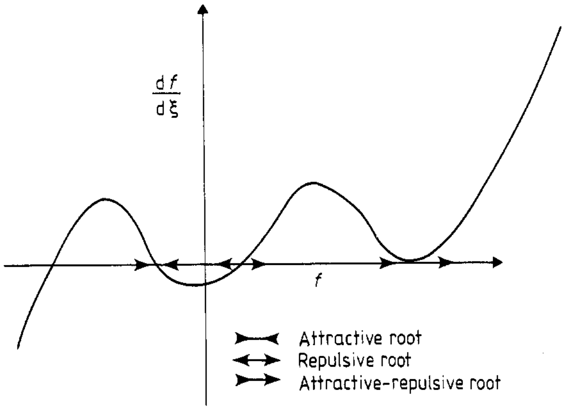

Figure 4. A graph of $df/d\xi$ as a function of f for (14).

This can be plotted the same way as before. But this time the plot should be done for each value of a_0 separately. The result is a family of orbits defining the trajectories of real periodic solutions which satisfy the variational principle for the free energy in the configuration space. A schematic illustration of this case is shown in the lower part of figure 2. Each orbit has a different but finite energy density which progressively increases with the radius of the orbit for $d>0$ and decreases for $d<0$. Orbits corresponding to infinite approaches to a turning point have finite total energies.

7. Discussion and conclusions

In this paper we have presented a comprehensive analysis of the time-dependent Landau-Ginzburg equation in $(3+1)$ spacetime dimensions. The method used was the symmetry reduction for PDE. We have found three distinct cases: (i) the vicinity of the critical point; (ii) the vicinity of the tricritical point; (iii) the remaining regions of the phase diagram. Each of these cases leads to different reduced ODE and different symmetry variables although some of them appear to be special cases. We have either solved the obtained equations or referred the reader to appropriate papers where solutions have been published. Several cases do not satisfy the Painlevé property and it is not known if they can be solved analytically. We have found time-dependent homogeneous phases which interpolate between stable phases of the free energy functional, through various types of decay. We have also found time-independent, translationally; cylindrically or spherically symmetric solutions of a variety of functional forms which include real solutions with or without singularities, periodic and localised types. We have also found reductions (only at the critical or tricritical points) which lead to solutions which exhibit stationary spiral patterns and are radially damped from the z axis. These do appear similar to the so-called Winfree waves which have been discovered in the kinetics of chemical reactions, and described using the formalism of reaction–diffusion equations (Ortoleva 1980) analogous to our TDLG. In the past, theoretical description was provided by simply postulating the form of the symmetry variable. Here, we believe for the first time, we have derived it from symmetry principles. One reduction leads to a spherically symmetric wavefront which propagates with time. However, its analytic form is unknown. We have illustrated the reductions leading to non-Painlevé ODE with numerical and approximate solutions.

Finally, we wish to mention that in some of these cases one can calculate the energy or energy density of the solutions, depending on the case. First, for translationally invariant solutions this has been done already for $a = d = 0$ by Winternitz *et al* (1988a), who obtained explicit expressions for the energy per volume of all periodic solutions and the total energy of all localised solutions performing direct integration. The same method is currently applied to $d = 0$, $a \neq 0$ (Winternitz *et al* 1989a) and to $d \neq 0$; $a = 0$ (Winternitz *et al* 1989b). For cylindrically and spherically symmetric solutions the energy functional, (2) can be significantly simplified using the equation of motion, (37) or (40), respectively, and integrating by parts. The result is that for cylindrically symmetric solutions the energy density is

$$\lim_{v\to\infty}\frac{G}{V}=\lim_{R\to\infty}\left\{G_0(\eta(R))-\tfrac{1}{2}\delta\left(\frac{\mathrm{d}\eta}{\mathrm{d}\rho}(R)\right)^2+\delta\left[\int_0^R \mathrm{d}\rho\,\rho\left(\frac{\mathrm{d}\eta}{\mathrm{d}\rho}\right)^2\right]R^{-2}\right\} \tag{61}$$

and for spherically symmetric solutions

$$\lim_{v\to\infty}\frac{G}{V}=\lim_{R\to\infty}\left\{G_0(\eta(R))-\tfrac{1}{2}\delta\left(\frac{\mathrm{d}\eta}{\mathrm{d}\rho}(R)\right)^2+\delta\left[\int_0^R \rho^2\left(\frac{\mathrm{d}\eta}{\mathrm{d}\rho}\right)^2\mathrm{d}\rho\right]R^{-3}\right\}. \tag{62}$$

Note that in both cases the bracketed terms depend only on the boundary conditions while the second terms depend on the bulk properties of η. For this reason, if η has a singularity at the origin or at infinity, the energy density of the solution will also be divergent. If, on the other hand, there is a singularity at $0<\rho_0<\infty$, then, for $\eta \sim A(\rho-\rho_0)^{-\nu}$ being the type of singularity at ρ_0, the energy density will: (a) diverge if $\nu>\frac{1}{2}$ (respectively 1) for cylindrical (respectively spherical) solutions; (b) be unaffected by the singularity if $\nu<\frac{1}{2}$ (respectively 1); (c) be changed by a constant equal to $-(A\nu)^2/(2\nu+1)$ if $\nu=\frac{1}{2}$ (respectively 1). Calculations of the energies of the remaining solutions are underway. We believe that both spherically and cylindrically symmetric solutions are very important at criticality, especially in liquid crystals, fluids and gases where rotational symmetries are manifest and spherical or cylindrical boundaries are most natural. Thus, further analyses of the properties of these solutions should be of great relevance to the kinetics of phase transitions for these types of materials.

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